$\mathcal{D}$-ELLIPTIC SHEAVES AND ODD JACOBIANS

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Abstract. We examine the existence of rational divisors on modular curves of $\mathcal{D}$-elliptic sheaves and on Atkin-Lehner quotients of these curves over local fields. Using a criterion of Poonen and Stoll, we show that in infinitely many cases the Tate-Shafarevich groups of the Jacobians of these Atkin-Lehner quotients have non-square orders.

1. Introduction

Let F is a global field. Let C be a smooth projective geometrically irreducible curve of genus g over F. Denote by $|F|$ the set of places of F. For $x \in |F|$, denote by F_x the completion of F at x. A place $x \in |F|$ is called *deficient* for C if $C_{F_x} := C \times_F F_x$ has no F_x-rational divisors of degree $g-1$, cf. [19]. It is known that the number of deficient places is finite. Let J be the Jacobian variety of C. Assume the Tate-Shafarevich group $Ш(J)$ is finite. In [19], Poonen and Stoll show that the order of $Ш(J)$ can be a square as well as twice a square. In the first case J is called *even*, and in the second case J is called *odd*. The parity of the number of deficient places is directly related to the parity of J [19, Section 8]:

Theorem 1.1. *J is even if and only if the number of deficient places for C is even.*

Using this theorem, Poonen and Stoll show that infinitely many hyperelliptic Jacobians over $\mathbb{Q}$ are odd for every even genus. Moreover, for certain explicit genus 2 and 3 curves over $\mathbb{Q}$ they are able to prove that $Ш(J)$ is finite and has non-square order. In [9], applying Theorem 1.1 to quotients of Shimura curves under the action of Atkin-Lehner involutions, Jordan and Livné show that infinitely many of these quotient curves have odd Jacobians.

For function fields, Proposition 30 in [19] gives the following example: Let J be the Jacobian of the genus 2 curve

$$C : y^2 = Tx^6 + x - aT$$

over $\mathbb{F}_q(T)$, where q is odd, and $a \in \mathbb{F}_q^\times$ is a non-square. As one checks, only the place $\infty = 1/T$ is deficient for C. Next, as is observed in [19, p. 1141], C defines a rational surface over $\mathbb{F}_q$, so the Brauer group of that surface is finite. The main theorem in [6] then implies that $Ш(J)$ is also finite. Overall, $Ш(J)$ is finite and has non-square order.

In this paper we adapt the idea of Jordan and Livné [9] to $\mathbb{F}_q(T)$, and exhibit infinitely many curves over $\mathbb{F}_q(T)$ whose Jacobians are odd. These curves are obtained as quotients of modular curves of $\mathcal{D}$-elliptic sheaves under the action of Atkin-Lehner involutions. We also show that only finitely many of these curves

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can be hyperelliptic, and in some cases prove that the Tate-Shafarevich groups in question are indeed finite.

2. Notation and terminology

2.1. **Notation.** Let $F = \mathbb{F}_q(T)$ be the field of rational functions on the projective line $\mathbb{P} := \mathbb{P}^1_{\mathbb{F}_q}$ over the finite field $\mathbb{F}_q$. Fix the place $\infty := 1/T$. For $x \in |F|$, denote by F_x and $\mathcal{O}_x$ the completion of F and $\mathcal{O}_{\mathbb{P},x}$ at x, respectively. The residue field of $\mathcal{O}_x$ is denoted by $\mathbb{F}_x$ and the cardinality of $\mathbb{F}_x$ is denoted by q_x. The degree of x is $\deg(x) := [\mathbb{F}_x : \mathbb{F}_q]$. Let ϖ_x be a uniformizer of $\mathcal{O}_x$. We assume that the valuation $\mathrm{ord}_x : F_x \to \mathbb{Z}$ is normalized by $\mathrm{ord}_x(\varpi_x) = 1$. Let $A = \mathbb{F}_q[T]$ be the polynomial ring over $\mathbb{F}_q$; this is the subring of F consisting of functions which are regular away from ∞. For a place $x \neq \infty$, let $\mathfrak{p}_x$ be the corresponding prime ideal of A, and $\wp_x \in A$ be the monic generator of $\mathfrak{p}_x$.

For a ring H with a unit element, we denote by $H^\times$ the group of its invertible elements.

For $S \subset |F|$, put

$$\mathrm{Odd}(S) = \begin{cases} 1, & \text{if all places in } S \text{ have odd degrees;} \\ 0, & \text{otherwise.} \end{cases}$$

2.2. **Quaternion algebras.** Let D be a *quaternion algebra* over F, i.e., a 4-dimensional F-algebra with center F which does not possess non-trivial two-sided ideals. Denote $D_x := D \otimes_F F_x$; this is a quaternion algebra over F_x. By Wedderburn's theorem [20, (7.4)], a quaternion algebra is either a division algebra or is isomorphic to the algebra of 2×2 matrices. We say that the algebra D *ramifies* (resp. *splits*) at $x \in |F|$ if D_x is a division algebra (resp. $D_x \cong \mathbb{M}_2(F_x)$). Let $R \subset |F|$ be the set of places where D ramifies. It is known that R is a finite set of even cardinality, and for any choice of a finite set $R \subset |F|$ of even cardinality there is a unique, up to isomorphism, quaternion algebra ramified exactly at the places in R; see [28, p. 74]. In particular, $D \cong \mathbb{M}_2(F)$ if and only if $R = \emptyset$. We denote the *reduced norm* of $\alpha \in D$ by $\mathrm{Nr}(\alpha)$; for the definition see [20, (9.6a)].

An $\mathcal{O}_{\mathbb{P}}$*-order* in D is a sheaf of $\mathcal{O}_{\mathbb{P}}$-algebras with generic fibre D which is coherent and locally free as an $\mathcal{O}_{\mathbb{P}}$-module. A $\mathcal{D}$*-bimodule* for an $\mathcal{O}_{\mathbb{P}}$-order $\mathcal{D}$ in D is an $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{I}$ with left and right $\mathcal{D}$-actions compatible with the $\mathcal{O}_{\mathbb{P}}$-action and such that

$$(\lambda i)\mu = \lambda(i\mu), \quad \text{for any } \lambda, \mu \in \mathcal{D} \text{ and } i \in \mathcal{I}.$$

A $\mathcal{D}$-bimodule $\mathcal{I}$ is *invertible* if there is another $\mathcal{D}$-bimodule $\mathcal{J}$ such that there are isomorphism of $\mathcal{D}$-bimodules

$$\mathcal{I} \otimes_{\mathcal{D}} \mathcal{J} \cong \mathcal{D}, \quad \mathcal{J} \otimes_{\mathcal{D}} \mathcal{I} \cong \mathcal{D}.$$

The group of isomorphism classes of invertible $\mathcal{D}$-bimodules will be denoted by $\mathrm{Pic}(\mathcal{D})$: the group operation is $\mathcal{I}_1 \otimes_{\mathcal{D}} \mathcal{I}_2$, cf. [20, (37.5)].

2.3. **Graphs.** We recall some of the terminology related to graphs, as presented in [25] and [11]. A graph $\mathcal{G}$ consists of a set of *vertices* $\mathrm{Ver}(\mathcal{G})$ and a set of *edges* $\mathrm{Ed}(\mathcal{G})$. Every edge y has *origin* $o(y) \in \mathrm{Ver}(\mathcal{G})$, *terminus* $t(y) \in \mathrm{Ver}(\mathcal{G})$, and *inverse* edge $\bar{y} \in \mathrm{Ed}(\mathcal{G})$ such that $\bar{\bar{y}} = y$ and $o(y) = t(\bar{y})$, $t(y) = o(\bar{y})$. The vertices $o(y)$ and $t(y)$ are the *extremities* of y. Note that it is allowed for distinct edges $y \neq z$ to have $o(y) = o(z)$ and $t(y) = t(z)$. We say that two vertices are *adjacent* if they are

the extremities of some edge. The graph $\mathcal{G}$ is a *graph with lengths* if we are given a map

$$\ell = \ell_{\mathcal{G}} : \mathrm{Ed}(\mathcal{G}) \to \mathbb{N} = \{1, 2, 3, \cdots\}$$

such that $\ell(y) = \ell(\bar{y})$. An *automorphism* of $\mathcal{G}$ is a pair $\phi = (\phi_1, \phi_2)$ of bijections $\phi_1 : \mathrm{Ver}(\mathcal{G}) \to \mathrm{Ver}(\mathcal{G})$ and $\phi_2 : \mathrm{Ed}(\mathcal{G}) \to \mathrm{Ed}(\mathcal{G})$ such that $\phi_1(o(y)) = o(\phi_2(y))$, $\overline{\phi_2(y)} = \phi_2(\bar{y})$, and $\ell(y) = \ell(\phi_2(y))$.

Let Γ be a group acting on a graph $\mathcal{G}$ (i.e., Γ acts via automorphisms). For $v \in \mathrm{Ver}(\mathcal{G})$, denote

$$\mathrm{Stab}_{\Gamma}(v) = \{\gamma \in \Gamma \mid \gamma v = v\}$$

the stabilizer of v in Γ. Similarly, let $\mathrm{Stab}_{\Gamma}(y) = \mathrm{Stab}_{\Gamma}(\bar{y})$ be the stabilizer of $y \in \mathrm{Ed}(\mathcal{G})$. There is a quotient graph $\Gamma \setminus \mathcal{G}$ such that $\mathrm{Ver}(\Gamma \setminus \mathcal{G}) = \Gamma \setminus \mathrm{Ver}(\mathcal{G})$ and $\mathrm{Ed}(\Gamma \setminus \mathcal{G}) = \Gamma \setminus \mathrm{Ed}(\mathcal{G})$.

2.4. **Mumford uniformization.** Let $\mathcal{O}$ be a complete discrete valuation ring with fraction field K, finite residue field k and a uniformizer π. Let Γ be a subgroup of $\mathrm{GL}_2(K)$ whose image $\overline{\Gamma}$ in $\mathrm{PGL}_2(K)$ is discrete with compact quotient. There is a formal scheme $\widehat{\Omega}$ over $\mathrm{Spf}(\mathcal{O})$ which is equipped with a natural action of $\mathrm{PGL}_2(K)$ and parametrizes certain formal groups. Kurihara in [11] extended Mumford's fundamental result [14] and proved the following: there is a normal, proper and flat scheme X^{Γ} over $\mathrm{Spec}(\mathcal{O})$ such that the formal completion of X^{Γ} along its closed fibre is isomorphic to the quotient $\Gamma \setminus \widehat{\Omega}$. The generic fibre X^{Γ}_K is a smooth, geometrically integral curve over K. The closed fibre X^{Γ}_k is reduced with normal crossing singularities, and every irreducible component is isomorphic to $\mathbb{P}^1_k$. If x is a double point on X^{Γ}_k, then there exists a unique integer m_x for which the completion of $\mathcal{O}_{x,X} \otimes_{\mathcal{O}} \widehat{\mathcal{O}}^{\mathrm{ur}}$ is isomorphic to the completion of $\widehat{\mathcal{O}}^{\mathrm{ur}}[t,s]/(ts - \pi^{m_x})$. Here $\widehat{\mathcal{O}}^{\mathrm{ur}}$ denotes the completion of the maximal unramified extension of $\mathcal{O}$.

Remark 2.1. $\widehat{\Omega}$ is the formal scheme associated to Drinfeld's non-archimedean half-plane $\Omega = \mathbb{P}^{1,\mathrm{an}}_K - \mathbb{P}^{1,\mathrm{an}}_K(K)$ over K. For the description of the rigid-analytic structure of Ω and the construction of $\widehat{\Omega}$ we refer to Chapter I in [2].

The *dual graph* $\mathcal{G}$ of X^{Γ} is the following graph with lengths. The vertices of $\mathcal{G}$ are the irreducible components of X^{Γ}_k. The edges of $\mathcal{G}$, ignoring the orientation, are the singular points of X^{Γ}_k. If x is a double point and $\{y, \bar{y}\}$ is the corresponding edge of $\mathcal{G}$, then the extremities of y and $\bar{y}$ are the irreducible components passing through x; choosing between y or $\bar{y}$ corresponds to choosing one of the branches through x. Finally, $\ell(y) = \ell(\bar{y}) = m_x$.

Let $\mathcal{T}$ be the graph whose vertices $\mathrm{Ver}(\mathcal{T}) = \{[\Lambda]\}$ are the homothety classes of $\mathcal{O}$-lattices in K^2, and two vertices $[\Lambda]$ and $[\Lambda']$ are adjacent if we can choose representatives $L \in [\Lambda]$ and $L' \in [\Lambda']$ such that $L' \subset L$ and $L/L' \cong k$. One shows that $\mathcal{T}$ is an infinite tree in which every vertex is adjacent to exactly $\#k + 1$ other vertices. This is the *Bruhat-Tits tree* of $\mathrm{PGL}_2(K)$, cf. [25, p. 70]. The group $\mathrm{GL}_2(K)$ acts on $\mathcal{T}$ as the group of linear automorphisms of K^2, so the group Γ also acts on $\mathcal{T}$. We assign lengths to the edges of the quotient graph $\Gamma \setminus \mathcal{T}$: for $y \in \mathrm{Ed}(\Gamma \setminus \mathcal{T})$ let $\ell(y) = \#\mathrm{Stab}_{\overline{\Gamma}}(\tilde{y})$, where $\tilde{y}$ is a preimage of y in $\mathcal{T}$. By Proposition 3.2 in [11], there is an isomorphism $\mathcal{G} \cong \Gamma \setminus \mathcal{T}$ of graphs with lengths.

Notation 2.2. For $x \in |F|$, we denote Mumford's formal scheme over $\mathrm{Spf}(\mathcal{O}_x)$ by $\widehat{\Omega}_x$, and the Bruhat-Tits tree of $\mathrm{PGL}_2(F_x)$ by $\mathcal{T}_x$.

3. Modular curves of $\mathcal{D}$-elliptic sheaves

3.1. **$\mathcal{D}$-elliptic sheaves.** The notion of $\mathcal{D}$-elliptic sheaves was introduced in [12]. Here we follow [26], which gives a somewhat different (but equivalent) definition of $\mathcal{D}$-elliptic sheaves that is more convenient for our purposes.

From now on we assume that D is a division quaternion algebra which is split at ∞. Let $\mathcal{D}$ be an $\mathcal{O}_{\mathbb{P}}$-order in D such that $\mathcal{D}_x := \mathcal{D}\otimes_{\mathcal{O}_{\mathbb{P}}}\mathcal{O}_x$ is a maximal order in D_x for any $x \neq \infty$, and $\mathcal{D}_\infty$ is isomorphic to the subring of $\mathbb{M}_2(\mathcal{O}_\infty)$ which are upper triangular modulo ϖ_∞. Let $\mathcal{D}(-\frac{1}{2}\infty)$ denote the two-sided ideal in $\mathcal{D}$ given by $\mathcal{D}(-\frac{1}{2}\infty)_x = \mathcal{D}_x$ for all $x\neq\infty$, and $\mathcal{D}(-\frac{1}{2}\infty)_\infty$ is the radical of $\mathcal{D}_x$. Concretely, $\mathcal{D}(-\frac{1}{2}\infty)_\infty$ is the ideal of $\mathcal{D}_\infty$ consisting of matrices which are upper triangular modulo ϖ_∞ with zeros on the diagonal.

Definition 3.1. A *$\mathcal{D}$-elliptic sheaf with pole* ∞ over an $\mathbb{F}_q$-scheme S is a pair $E = (\mathcal{E}, t)$ consisting of a locally free right $\mathcal{D}\boxtimes\mathcal{O}_S$-module of rank 1 and an injective homomorphism of $\mathcal{D}\boxtimes\mathcal{O}_S$-modules

$$t : (\mathrm{id}_{\mathbb{P}}\times\mathrm{Frob}_S)^*(\mathcal{E}\otimes_{\mathcal{D}}\mathcal{D}(-\frac{1}{2}\infty)) \to \mathcal{E}$$

such that the cokernel of t is supported on the graph $\Gamma_z \subset \mathbb{P}\times_{\mathrm{Spec}\mathbb{F}_q} S$ of a morphism $z : S \to \mathbb{P}$ and is a locally free $\mathcal{O}_S$-module of rank 2.

Remark 3.2. In [26], this definition is given for an arbitrary central simple algebra D over an arbitrary function field F. Moreover, the order $\mathcal{D}$ is only assumed to be hereditary.

Theorem 3.3. *The moduli stack of $\mathcal{D}$-elliptic sheaves of fixed degree* $\deg(\mathcal{E}) = -1$ *admits a coarse moduli scheme X^R. The canonical morphism $X^R \to \mathbb{P}$ is projective with geometrically irreducible fibres of pure relative dimension* 1*, and it is smooth over $\mathbb{P} - R - \infty$.*

Proof. This theorem follows from one of the main results in [12]; cf. [26, §4.3]. □

The genus of the curve X^R_F is given by the formula (see [16])

$$g(X^R) = 1 + \frac{1}{q^2-1}\prod_{x\in R}(q_x - 1) - \frac{q}{q+1}\cdot 2^{\#R-1}\cdot \mathrm{Odd}(R). \tag{3.1}$$

3.2. **Atkin-Lehner involutions.** Let $\mathfrak{P}_x$ be the radical of $\mathcal{D}_x$. By [20, (39.1)], $\mathfrak{P}_x$ is a two-sided ideal in $\mathcal{D}_x$, and every two-sided ideal of $\mathcal{D}_x$ is an integral power of $\mathfrak{P}_x$. It is known that there exists $\Pi_x \in \mathfrak{P}_x$ such that $\Pi_x\mathcal{D}_x = \mathcal{D}_x\Pi_x = \mathfrak{P}_x$. The positive integer e_x such that $\mathfrak{P}_x^{e_x} = \varpi_x\mathcal{D}_x$ is the *index* of $\mathcal{D}_x$. With this definition, $e_x = 2$ if $x \in R\cup\infty$, and $e_x = 1$, otherwise. Define the group of divisors

$$\mathrm{Div}(\mathcal{D}) := \left\{ \sum_{x\in|F|} n_x x \in \bigoplus_{x\in|F|}\mathbb{Q}x \;\middle|\; e_x n_x \in \mathbb{Z} \text{ for any } x\in|F| \right\}.$$

For a divisor $Z = \sum_{x\in|F|} n_x x \in \mathrm{Div}(\mathcal{D})$, let $\mathcal{D}(Z)$ be the invertible $\mathcal{D}$-bimodule given by $\mathcal{D}(Z)|_{\mathbb{P}-\mathrm{Supp}(Z)} = \mathcal{D}|_{\mathbb{P}-\mathrm{Supp}(Z)}$ and $\mathcal{D}(Z)_x = \mathfrak{P}_x^{-n_x e_x}$ for all $x\in\mathrm{Supp}(Z)$. For each $f \in F^\times$ there is an associated divisor $\mathrm{div}(f) = \sum_{x\in|F|}\mathrm{ord}_x(f)x$, which we consider as an element of $\mathrm{Div}(\mathcal{D})$. It follows from [20, (40.9)] that the sequence

$$0 \to F^\times/\mathbb{F}_q^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\mathcal{D}) \xrightarrow{Z\mapsto\mathcal{D}(Z)} \mathrm{Pic}(\mathcal{D}) \to 0 \tag{3.2}$$

is exact, cf. [26, §3.2]. Let $\mathrm{Div}^0(\mathcal{D}) \subset \mathrm{Div}(\mathcal{D})$ be the subgroup of degree 0 divisors: $\sum_{x\in|F|} n_x x \in \mathrm{Div}^0(\mathcal{D})$ if $\sum_{x\in|F|} n_x \deg(x) = 0$. Define $\mathrm{Pic}^0(\mathcal{D})$ to be the image of $\mathrm{Div}^0(\mathcal{D})$ in $\mathrm{Pic}(\mathcal{D})$. It is easy to check that $\mathrm{Pic}^0(\mathcal{D}) \cong (\mathbb{Z}/2\mathbb{Z})^{\#R}$, and is generated by the divisors $\left(\frac{\deg(x)}{2}\infty - \frac{1}{2}x\right)$, $x \in R$.

If $\mathcal{L} \in \mathrm{Pic}(\mathcal{D})$, then

$$E = (\mathcal{E}, t) \mapsto E \otimes \mathcal{L} := (\mathcal{E} \otimes_{\mathcal{D}} \mathcal{L}, t \otimes_{\mathcal{D}} \mathrm{id}_{\mathcal{L}})$$

defines an automorphism of the stack of $\mathcal{D}$-elliptic sheaves. Moreover, if $\mathcal{L} \in \mathrm{Pic}^0(\mathcal{D})$, then this action preserves the substack consisting of $(\mathcal{E}, t)$ with $\deg(\mathcal{E})$ fixed, cf. [26, §4.1]. Hence $W := \mathrm{Pic}^0(\mathcal{D})$ acts on X^R by automorphisms.

Definition 3.4. We call the subgroup W of $\mathrm{Aut}(X^R)$ the *group of Atkin-Lehner involutions*, and we denote by $w_x \in W$, $x \in R$, the automorphism induced by $\mathcal{D}\left(\frac{\deg(x)}{2}\infty - \frac{1}{2}x\right)$.

Remark 3.5. It follows from [17, Thm. 4.6] that if $\mathrm{Odd}(R) = 0$, then $\mathrm{Aut}(X^R) = W$.

The *normalizer* of $\mathcal{D}_x$ in D_x is the subgroup of $D_x^\times$

$$N(\mathcal{D}_x) = \{g \in D_x^\times \mid g\mathcal{D}_x g^{-1} = \mathcal{D}_x\}.$$

If $g \in N(\mathcal{D}_x)$, then $g\mathcal{D}_x$ is a two-sided ideal of $\mathcal{D}_x$, so there exists $m \in \mathbb{Z}$ such that $g\mathcal{D}_x = \mathfrak{P}_x^m$ Define $v_{\mathcal{D}_x}(g) = \frac{m}{e_x}$. Note that for $g \in F_x \subset N(\mathcal{D}_x)$, we have $\mathrm{ord}_x(g) = v_{\mathcal{D}_x}(g)$.

Let $\mathcal{C}(\mathcal{D}) := \prod'_{x\in|F|} N(\mathcal{D}_x)/F^\times \prod_{x\in|F|} \mathcal{D}_x^\times$, where $\prod'_{x\in|F|} N(\mathcal{D}_x)$ denotes the restricted direct product of the groups $\{N(\mathcal{D}_x)\}_{x\in|F|}$ with respect to $\{\mathcal{D}_x^\times\}_{x\in|F|}$. Given $a = \{a_x\}_x \in \prod'_{x\in|F|} N(\mathcal{D}_x)$, we put $\mathrm{div}(a) = \sum_{x\in|F|} v_{\mathcal{D}_x}(a_x)x$. The assignment $a \mapsto \mathcal{D}(\mathrm{div}(a))$ induces an isomorphism [26, Cor. 3.4]:

$$\mathcal{C}(\mathcal{D}) \cong \mathrm{Pic}(\mathcal{D}). \tag{3.3}$$

Let $\mathcal{D}^\infty := H^0(\mathbb{P} - \infty, \mathcal{D})$; this is a maximal A-order in D. Let $\Gamma^\infty := (\mathcal{D}^\infty)^\times$ be the units in $\mathcal{D}^\infty$. Define the *normalizer* of $\mathcal{D}^\infty$ in D as

$$N(\mathcal{D}^\infty) := \{g \in D^\times \mid g\mathcal{D}^\infty g^{-1} = \mathcal{D}^\infty\}.$$

Denote $\mathcal{C}(\mathcal{D}^\infty) = N(\mathcal{D}^\infty)/F^\times\Gamma^\infty$. Then (3.3) induces an isomorphism

$$\mathcal{C}(\mathcal{D}^\infty) \cong \mathrm{Pic}^0(\mathcal{D}). \tag{3.4}$$

By (37.25) and (37.28) in [20], the natural homomorphism

$$N(\mathcal{D}^\infty)/F^\times\Gamma^\infty \to \prod_{x\in|F|-\infty} N(\mathcal{D}_x)/F_x^\times\mathcal{D}_x^\times \tag{3.5}$$

is an isomorphism. Next, by (37.26) and (37.27) in [20],

$$N(\mathcal{D}_x)/F_x^\times\mathcal{D}_x^\times \cong \begin{cases} 1, & \text{if } x \notin R \cup \infty; \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } x \in R. \end{cases}$$

For $x \in R$, the non-trivial element of $N(\mathcal{D}_x)/F_x^\times\mathcal{D}_x^\times$ is the image of Π_x. According to [20, (34.8)], there exist elements $\{\lambda_x \in \mathcal{D}^\infty\}_{x\in R}$ such that $\mathrm{Nr}(\lambda_x)A = \mathfrak{p}_x$. The image of λ_x in $\mathcal{D}_x$ can be taken as Π_x. Overall, $\mathcal{C}(\mathcal{D}^\infty) \cong (\mathbb{Z}/2Z)^{\#R}$ is generated by λ_x's, and the isomorphism (3.4) is given by $w_x \mapsto \lambda_x$.

3.3. **Uniformization theorems.** Since $D_\infty \cong \mathbb{M}_2(F_\infty)$, the group Γ^∞ can be considered as a discrete cocompact subgroup of $\mathrm{GL}_2(F_\infty)$ via an embedding

$$\Gamma^\infty \hookrightarrow D^\times(F_\infty) \cong \mathrm{GL}_2(F_\infty).$$

Let $\widehat{X}^R_{\mathcal{O}_\infty}$ denote the completion of $X^R_{\mathcal{O}_\infty}$ along its special fibre. By a theorem of Blum and Stuhler [1, Thm. 4.4.11], there an isomorphism of formal $\mathcal{O}_\infty$-schemes

$$\Gamma^\infty \setminus \widehat{\Omega}_\infty \cong \widehat{X}^R_{\mathcal{O}_\infty}, \tag{3.6}$$

which is compatible with the action of W; see [26, §4.6]. More precisely, the action of w_x on $\Gamma^\infty \setminus \widehat{\Omega}_\infty$ induced by (3.6) is given by the action of λ_x considered as an element of $\mathrm{GL}_2(F_\infty)$. Note that λ_x is in the normalizer of Γ^∞, so it acts on the quotient $\Gamma^\infty \setminus \widehat{\Omega}_\infty$ and this action does not depend on a particular choice of λ_x.

Now fix some $x \in R$. Let $\bar{D}$ be the quaternion algebra over F which is ramified exactly at $(R-x)\cup\infty$. Fix a maximal A-order $\mathfrak{D}$ in $\bar{D}(F)$, and denote

$$\begin{aligned} &A^x = A[\wp_x^{-1}];\\ &\mathfrak{D}^x = \mathfrak{D}\otimes_A A^x;\\ &\mathfrak{D}^{x,2} = \{\gamma \in \mathfrak{D}^x \mid \mathrm{ord}_x(\mathrm{Nr}(\gamma)) \in 2\mathbb{Z}\};\\ &\Gamma^x = (\mathfrak{D}^{x,2})^\times. \end{aligned}$$

If we fix an identification of $\bar{D}_x$ with $\mathbb{M}_2(F_x)$, then Γ^x is a subgroup of $\mathrm{GL}_2(F_x)$ whose image $\Gamma^x/(A^x)^\times$ in $\mathrm{PGL}_2(F_x)$ is discrete and cocompact. Let $\mathcal{O}_x^{(2)}$ be the unramified quadratic extension of $\mathcal{O}_x$. Let $\gamma_x \in \mathfrak{D}^x$ be an element such that $\mathrm{Nr}(\gamma_x)A = \mathfrak{p}_x$. Such γ_x exists by [20, (34.8)] and it normalizes Γ^x, hence acts on $\Gamma^x \setminus \widehat{\Omega}_x$. Let $\widehat{X}^R_{\mathcal{O}_x}$ denote the completion of $X^R_{\mathcal{O}_x}$ along its special fibre. By the analogue of the Čerednik-Drinfeld uniformization, proven in this context by Hausberger [7], there is an isomorphism of formal $\mathcal{O}_x$-schemes

$$[(\Gamma^x \setminus \widehat{\Omega}_x)\otimes \mathcal{O}_x^{(2)}]/(\gamma_x \otimes \mathrm{Frob}_x^{-1}) \cong \widehat{X}^R_{\mathcal{O}_x}, \tag{3.7}$$

where $\mathrm{Frob}_x : \mathcal{O}_x^{(2)} \to \mathcal{O}_x^{(2)}$ denotes the lift of the Frobenius homomorphism $a \mapsto a^{q_x}$ on $\overline{\mathbb{F}}_x$ to an $\mathcal{O}_x$-homomorphism.

Let $N(\mathfrak{D}^{x,2})$ be the normalizer of $\mathfrak{D}^{x,2}$ in $\bar{D}$, and

$$\mathcal{C}(\mathfrak{D}^{x,2}) := N(\mathfrak{D}^{x,2})/F^\times\Gamma^x.$$

As in (3.5), the natural homomorphism

$$N(\mathfrak{D}^{x,2})/F^\times\Gamma^x \to \prod_{y\in|F|-\infty} N(\mathfrak{D}^{x,2}_y)/F_y^\times(\mathfrak{D}^{x,2}_y)^\times$$

is an isomorphism. The normalizer $N(\mathfrak{D}^{x,2}_x)$ is $F_x^\times(\mathfrak{D}^x_x)^\times$, so we have

$$N(\mathfrak{D}^{x,2}_x)/F_x^\times(\mathfrak{D}^{x,2}_x)^\times \cong \mathbb{Z}/2\mathbb{Z},$$

generated by γ_x. On the other hand, if $y \neq x$, then

$$N(\mathfrak{D}^{x,2}_y)/F_y^\times(\mathfrak{D}^{x,2}_y)^\times \cong N(\mathcal{D}_y)/F_y^\times\mathcal{D}_y^\times.$$

We see that

$$\mathcal{C}(\mathfrak{D}^{x,2}) \cong (\mathbb{Z}/2\mathbb{Z})^{\#R},$$

generated by a set of elements $\{\gamma_y \in \mathfrak{D}^x\}_{y\in R}$ such that $\mathrm{Nr}(\gamma_y)A = \mathfrak{p}_y$. The group W is canonically isomorphic with $\mathcal{C}(\mathfrak{D}^{x,2})$ via $w_y \mapsto \gamma_y$. The isomorphism (3.7) is

compatible with the action of W: for $y \in R$, the action of w_y on the left hand-side of (3.7) is given by γ_y; see [26, §4.6].

4. Main results

Proposition 4.1. *Denote by $\mathrm{Div}^d_{F_x}(X^R)$ the set of Weil divisors on $X^R_{F_x}$ which are rational over F_x and have degree d.*

1. *If $x \notin R$, then $\mathrm{Div}^d_{F_x}(X^R) \neq \emptyset$ for any d.*
2. *If $x \in R$, then $\mathrm{Div}^d_{F_x}(X^R) \neq \emptyset$ for even d, and $\mathrm{Div}^d_{F_v}(X^R) = \emptyset$ for odd d.*

Proof. For $n \geq 1$, denote by $\mathbb{F}_x^{(n)}$ the degree n extension of $\mathbb{F}_x$, and by $F_x^{(n)}$ the degree n unramified extension of F_x.

First, suppose $x \notin R \cup \infty$. By Theorem 3.3, $X^R_{\mathbb{F}_x}$ is a smooth projective curve. Weil's bound on the number of rational points on a curve over a finite field guarantees the existence of an integer $N \geq 1$ such that $X^R_{\mathbb{F}_x}(\mathbb{F}_x^{(n)}) \neq \emptyset$ for any $n \geq N$. The geometric version of Hensel's lemma [8, Lem. 1.1] implies that $X^R_{F_x}(F_x^{(n)}) \neq \emptyset$. Let $P \in X^R_{F_x}(F_x^{(N+1)})$ and $Q \in X^R_{F_x}(F_x^{(N)})$. The divisor $d \cdot Z$, where

$$Z = \sum_{\sigma \in \mathrm{Gal}(F_x^{(N+1)}/F_x)} P^\sigma - \sum_{\tau \in \mathrm{Gal}(F_x^{(N)}/F_x)} Q^\tau,$$

is F_x-rational and has degree d.

Next, suppose $x = \infty$. By (3.6), $X^R_{F_\infty}$ is Mumford uniformizable. This implies that $X^R_{F_\infty}$ has a regular model over $\mathcal{O}_\infty$ whose special fibre consists of $\mathbb{F}_\infty$-rational $\mathbb{P}^1$'s crossing at $\mathbb{F}_\infty$-rational points. In particular, over any extension $\mathbb{F}_\infty^{(n)}$, $n \geq 2$, there are smooth $\mathbb{F}_\infty^{(n)}$-rational points. Again by Hensel's lemma [8, Lem. 1.1], there are $F_\infty^{(n)}$-rational points on $X^R_{F_\infty}$ for any $n \geq 2$. The trace to F_∞ of such a point is in $\mathrm{Div}^n_{F_\infty}(X^R)$. One obtains a rational divisor of degree 1 by taking the difference of degree 3 and 2 rational divisors. This proves (1).

Finally, suppose $x \in R$. By [15, Thm. 4.1], $X^R_{F_x}(F_x^{(2)}) \neq \emptyset$. Taking the trace of an $F_x^{(2)}$-rational point and multiplying the resulting divisor by n, we see that $\mathrm{Div}^{2n}_{F_x}(X^R) \neq \emptyset$ for any n. Now suppose d is odd but $\mathrm{Div}^d_{F_x}(X^R) \neq \emptyset$. Let $Z \in \mathrm{Div}^d_{F_x}(X^R)$. Write $Z = Z_1 - Z_2$, where Z_1 and Z_2 are effective divisors. Since $\deg(Z) = \deg(Z_1) - \deg(Z_2)$ is odd, exactly one of these divisors has odd degree. Denote by F_x^{alg} the algebraic closure of F_x, F_x^{sep} the separable closure of F_x, and let $G := \mathrm{Gal}(F_x^{\mathrm{sep}}/F_x)$. Since Z is F_x-rational, both Z_1 and Z_2 are G-invariant. Assume without loss of generality that $\deg(Z_1)$ is odd. Write $Z_1 = Z_o + Z_e$, where $Z_o = \sum_{P \in X^R_{F_x}(F_x^{\mathrm{alg}})} n_P P$, $n_P \in \mathbb{Z}$ are odd, and $Z_e = \sum_{Q \in X^R_{F_x}(F_x^{\mathrm{alg}})} n_Q Q$, $n_Q \in \mathbb{Z}$ are even. Again Z_o and Z_e are G-invariant. Since $\deg(Z_e)$ is even, Z_o is non-zero. Since $\deg(Z_o)$ is necessarily odd, the support of Z_o must consist of an odd number of points. This set of points is G-invariant. We have a finite set of odd cardinality with an action of G, so one of the orbits necessarily has odd length. Thus, there is a point P in the support of Z such that the set of Galois conjugates of P has odd cardinality. This implies that the separable degree $[F_x(P) : F_x]_s$ is odd. If P is not separable, then the degree of inseparability of $F_x(P)$ over F_x divides the weight n_P of P in Z (as Z is F_x-rational). Since n_P is odd by assumption, the inseparable degree $[F_x(P) : F_x]_i$ is also odd. Overall, the degree of the extension $F_x(P)/F_x$ is

odd. We conclude that there is a finite extension K/F_x of odd degree such that $X^R_{F_x}(K) \neq \emptyset$. This contradicts [15, Thm. 4.1], so $\mathrm{Div}^d_{F_x}(X^R)$ must be empty. $\square$

Theorem 4.2. *Consider the following conditions:*

(1) *q is even;*
(2) *q is odd, $\#R = 2$, and $\mathrm{Odd}(R) = 1$.*

If one of these conditions holds, then the deficient places for X^R are the places in R. Otherwise, there are no deficient places for X^R. In either case, by Theorem 1.1, the Jacobian variety of X^R_F is even.

Proof. An elementary analysis of (3.1) shows that the genus $g(X^R)$ is even if and only if one of the above conditions holds. The claim of the theorem then follows from Proposition 4.1. $\square$

Next, we examine the existence of rational divisors on the quotients of X^R under the action of Atkin-Lehner involutions. For a fixed $y \in R$ we denote by $X^{(y)}$ the quotient curve X^R/w_y.

Proposition 4.3. *Denote by $\mathrm{Div}^d_{F_x}(X^{(y)})$ the set of Weil divisors on $X^{(y)}_{F_x}$ which are rational over F_x and have degree d.*

(1) *If $x \notin R$ or $x = y$, then $\mathrm{Div}^d_{F_x}(X^{(y)}) \neq \emptyset$ for any d.*
(2) *If $x \in R - y$ and d is even, then $\mathrm{Div}^d_{F_x}(X^{(y)}) \neq \emptyset$.*
(3) *If $x \in R - y$ and $\mathrm{Div}^d_{F_x}(X^{(y)}) \neq \emptyset$ for an odd d, then there is an extension K/F_x of odd degree such that $X^{(y)}_{F_x}(K) \neq \emptyset$.*

Proof. Since the Atkin-Lehner involutions are defined in terms of the moduli problem, the quotient morphism $\pi : X^R_{F_x} \to X^{(y)}_{F_x}$ is defined over F_x. Hence, if $Z \in \mathrm{Div}^d_{F_x}(X^R)$, then the pushforward $\pi_*(Z)$ is in $\mathrm{Div}^d_{F_x}(X^{(y)})$, so Proposition 4.1 implies (2) and (1) for $x \notin R$. Part (3) follows from the argument in the proof of Proposition 4.1. It remains to prove that $\mathrm{Div}^d_{F_y}(X^{(y)}) \neq \emptyset$ for any d. By (3.7) and ensuing discussion, $X^R_{F_y}$ is the $w_y \otimes \mathrm{Frob}_y^{-1}$ quadratic twist of the Mumford curve $\Gamma^y \setminus \widehat{\Omega}_y$. Hence the quotient $X^{(y)}_{F_y}$ of $X^R_{F_y}$ by w_y is Mumford uniformizable (without a twist) and one can argue as in the proof of Proposition 4.1 in the case when $x = \infty$. $\square$

Proposition 4.4. *Assume q is odd, and $x, y \in R$ are two distinct places of even degrees. If d is odd, then $\mathrm{Div}^{(d)}_{F_x}(X^{(y)}) = \emptyset$.*

Proof. Suppose d is odd and $\mathrm{Div}^{(d)}_{F_x}(X^{(y)}) \neq \emptyset$. Then by Proposition 4.3 there is an extension K/F_x of odd degree such that $X^{(y)}_{F_x}(K) \neq \emptyset$. The graph $G := \Gamma^x \setminus \mathcal{T}_x$ is the dual graph of the Mumford curve uniformized by Γ^x; see §2.4. From (3.7) we get an action of W on G. The same argument as in [9, p. 683] shows that if $X^{(y)}_{F_x}(K) \neq \emptyset$, then there is an edge s in G such that the following two conditions hold:

(1) either $\ell(s)$ is even or $w_y(s) = s$;
(2) either $w_x(s) = \bar{s}$ or $w_x w_y(s) = \bar{s}$.

By [15, Lem. 4.4], an edge of G has length 1 or $q+1$, and the number of edges of length $q+1$ is equal to

$$2^{\#R-1}\mathrm{Odd}(R-x)(1-\mathrm{Odd}(x)).$$

From the assumption that y has even degree we get that all edges of G have length 1. Thus, for the existence of K-rational points on $X_{F_x}^{(y)}$ we must have $w_y(s)=s$, and either $w_x(s)=\bar{s}$ or $w_xw_y(s)=\bar{s}$. Obviously $w_y(s)=s$ and $w_xw_y(s)=\bar{s}$ imply $w_x(s)=\bar{s}$. Therefore, the considerations reduce to a single case

$$w_x(s)=\bar{s} \quad \text{and} \quad w_y(s)=s.$$

Let $\tilde{s}$ be an edge of $\mathcal{T}_x$ lying above s. Modifying γ_x by an element of Γ^x, we may assume that $\gamma_x\tilde{s}=\bar{\tilde{s}}$. Let v be one of the extremities of $\tilde{s}$. Then γ_x^2 fixes v and $\mathrm{Nr}(\gamma_x^2)$ generates $\mathfrak{p}_x^2$. Thus, $\gamma_x^2\in F_x^\times\mu\mathrm{GL}_2(\mathcal{O}_x)\mu^{-1}$ for some $\mu\in\mathrm{GL}_2(F_x)$. By the norm condition, we get $\gamma_x^2=\wp_x c$, where $c\in\mu\mathrm{GL}_2(\mathcal{O}_x)\mu^{-1}$. Hence $\mathrm{ord}_x(\mathrm{Nr}(\gamma_x^2/\wp_x))=0$. On the other hand, since $\gamma_x^2/\wp_x$ also belongs to $\mathfrak{D}^x$, c belongs to a maximal A-order $\mathfrak{D}'$ in $\bar{D}$ (in fact, $\mathfrak{D}'=\mu\mathrm{GL}_2(\mathcal{O}_x)\mu^{-1}\cap\mathfrak{D}^x$). Since $\mathrm{Nr}(c)$ has zero valuation at every $v\in|F|-\infty$, $c\in(\mathfrak{D}')^\times$. By our assumption, $\deg(y)$ is even and $\bar{D}$ is ramified at y and ∞, so $(\mathfrak{D}')^\times\cong\mathbb{F}_q^\times$; cf. [4, Lem. 1]. Hence $\gamma_x^2=c\wp_x$, where $c\in\mathbb{F}_q^\times$. Since $\deg(x)$ is even, c must be a non-square, as otherwise ∞ splits in $F(\sqrt{c\wp_x})$, which contradicts the fact that this is a subfield of the quaternion algebra $\bar{D}$ ramified at ∞. Fix a non-square $\xi\in\mathbb{F}_q^\times$. Overall, we conclude that the condition $w_x(s)=\bar{s}$ translates into

$$\gamma_x^2=\xi\wp_x,$$

for an appropriate choice of γ_x.

Modifying γ_y by an element of Γ^x, we can further assume that $\gamma_y(\tilde{s})=\tilde{s}$. Next, note that γ_y belongs to some maximal A-order $\mathfrak{D}''$ in $\bar{D}$. Since $\bar{D}$ is ramified at y and $\mathrm{Nr}(\gamma_y)A=\mathfrak{p}_y$, the element γ_y generates the radical of $\mathfrak{D}_y^x$. Hence $\gamma_y^2=c\cdot\wp_y$, where $c\in\mathfrak{D}''$. Comparing the norms of both sides, we see that c must be a unit in $\mathfrak{D}''$. The same argument as with $\mathfrak{D}'$ shows that $(\mathfrak{D}'')^\times\cong\mathbb{F}_q^\times$, so after possibly scaling γ_y by a constant in $\mathbb{F}_q^\times$, we get

$$\gamma_y^2=\xi\wp_y.$$

Let $\langle\Gamma^x,\gamma_y\rangle$ be the subgroup of $\mathrm{GL}_2(F_x)$ generated by Γ^x and γ_y. By construction, the element γ_y fixes $\tilde{s}$. Since the edges of G have length 1, the stabilizer of $\tilde{s}$ in Γ^x is $(A^x)^\times$. Therefore,

$$\mathrm{Stab}_{\langle\Gamma^x,\gamma_y\rangle}(\tilde{s})/(A^x)^\times\subset\mathbb{F}_q(\gamma_y)^\times.$$

On the other hand, $\gamma_x^{-1}\gamma_y\gamma_x(\tilde{s})=\tilde{s}$. We conclude that there is $n\in\mathbb{Z}$ and $a,b\in\mathbb{F}_q$ (a,b are not both zero) such that

$$\gamma_y\gamma_x=\wp_x^n\gamma_x(a+b\gamma_y).$$

Now the same argument as in the proof of Part (3) of Theorem 4.1 in [15] shows that for such an equality to be true we must have $n=0$, $a=0$ and $b=-1$, i.e.,

$$\gamma_y\gamma_x=-\gamma_x\gamma_y.$$

The quadratic extensions $F(\gamma_x)$ and $F(\gamma_y)$ of F are obviously linearly disjoint. Therefore, $\bar{D}$ is isomorphic to the quaternion algebra $H(\xi\wp_x,\xi\wp_y)$ over F having the presentation:

$$i^2=\xi\wp_x, \quad j^2=\xi\wp_y, \quad ij=-ji.$$

As is well-known, the algebra $H(\xi\wp_x, \xi\wp_y)$ ramifies (resp. splits) at $v \in |F|$ if and only if the local symbol $(\xi\wp_x, \xi\wp_y)_v = -1$ (resp. $= 1$); cf. [28, p. 32]. On the other hand, by [24, p. 210]

$$(\xi\wp_x, \xi\wp_y)_x = \left(\frac{\xi\wp_y}{\mathfrak{p}_x}\right) \quad \text{and} \quad (\xi\wp_x, \xi\wp_y)_y = \left(\frac{\xi\wp_x}{\mathfrak{p}_y}\right),$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol. Since x and y have even degree, ξ is a square modulo $\mathfrak{p}_x$ and $\mathfrak{p}_y$. Thus, $\left(\frac{\xi\wp_y}{\mathfrak{p}_x}\right) = \left(\frac{\wp_y}{\mathfrak{p}_x}\right)$ and $\left(\frac{\xi\wp_x}{\mathfrak{p}_y}\right) = \left(\frac{\wp_x}{\mathfrak{p}_y}\right)$. The algebra $\bar{D}$ splits at x and ramifies at y, so we must have

$$\left(\frac{\wp_y}{\mathfrak{p}_x}\right) = 1 \quad \text{and} \quad \left(\frac{\wp_x}{\mathfrak{p}_y}\right) = -1.$$

But the quadratic reciprocity [21, Thm. 3.5] says that

$$\left(\frac{\wp_y}{\mathfrak{p}_x}\right)\left(\frac{\wp_x}{\mathfrak{p}_y}\right) = (-1)^{\frac{q-1}{2}\deg(x)\deg(y)} = 1.$$

This leads to a contradiction, so $\mathrm{Div}^d_{F_x}(X^{(y)}) = \emptyset$. $\square$

Theorem 4.5. *Assume q is odd and all places in R have even degrees. Consider the following three conditions:*

(1) *$R = \{x, y\}$, i.e., $\#R = 2$;*
(2) *$\left(\frac{\wp_y}{\mathfrak{p}_x}\right) = -1$;*
(3) *$\deg(y)$ is not divisible by 4.*

If one of these conditions fails, then there are no deficient places for $X^{(y)}$. If all three conditions hold, then x is the only deficient place for $X^{(y)}$. In the first case the Jacobian of $X^{(y)}_F$ is even and in the second case it is odd.

Proof. Let $\mathrm{Fix}(w_y)$ be the number of fixed points of w_y acting on X^R_F. By the Hurwitz genus formula applied to the quotient map $\pi : X^R_F \to X^{(y)}_F$, the genus of $X^{(y)}_F$ is equal to

$$g(X^{(y)}) = \frac{g(X^R) + 1}{2} - \frac{\mathrm{Fix}(w_y)}{4}.$$

(note that π has only tame ramification). On the other hand, by [17, Prop. 4.12]

$$\mathrm{Fix}(w_y) = h(\xi\wp_y) \prod_{x \in R} \left(1 - \left(\frac{\xi\wp_y}{\mathfrak{p}_x}\right)\right),$$

where $\xi \in \mathbb{F}_q^\times$ is a fixed non-square, and $h(\xi\wp_y)$ denotes the ideal class number of the Dedekind ring $\mathbb{F}_q[T, \sqrt{\xi\wp_y}]$. (A remark is in order: In [17], w_y is defined analytically as the involution of $\Gamma^\infty \setminus \Omega_\infty$ induced by λ_y, hence here we are implicitly using the fact that (3.6) is compatible with the action of W.) Combining these formulas, we get

$$g(X^{(y)}) = 1 + \frac{1}{2(q^2-1)} \prod_{x \in R} (q_x - 1) - \frac{h(\xi\wp_y)}{4} \prod_{x \in R} \left(1 - \left(\frac{\xi\wp_y}{\mathfrak{p}_x}\right)\right).$$

It is easy to see that the middle summand is always an even integer. Hence $g(X^{(y)})$ is even if and only if the last summand is odd. According to [3, Thm. 1], the class number $h(\xi\wp_y)$ is always even and it is divisible by 4 if and only if $\deg(y)$ is divisible by 4. Using this fact, one easily checks that the last summand is odd if and only if

the three conditions are satisfied. The theorem now follows from Propositions 4.3 and 4.4 $\square$

There are infinitely many pairs $R = \{x, y\}$ for which the conditions in Theorem 4.5 are satisfied. Indeed, fix an arbitrary y such that $\deg(y) \equiv 2 \pmod 4$. By the function field analogue of Dirichlet's theorem [21, Thm. 4.7], there are infinitely many places $x \in |F|$ of even degree such that $\left(\frac{\wp_x}{\mathfrak{p}_y}\right) = -1$. The quadratic reciprocity implies that for such places $\left(\frac{\wp_y}{\mathfrak{p}_x}\right) = -1$. Hence there are infinitely many $X_F^{(y)}$ with odd Jacobians.

Remark 4.6. For a fixed q there are only finitely many R such that $X_F^{(y)}$ is hyperelliptic. To see this, fix some $x \notin R \cup \infty$. Corollary 4.8 in [16] gives a lower bound on the number of $\mathbb{F}_x^{(2)}$-rational points on $X_{\mathbb{F}_x}^R$. Since the quotient map $X_{\mathbb{F}_x}^R \to X_{\mathbb{F}_x}^{(y)}$ is defined over $\mathbb{F}_x$ and has degree 2, from this bound we get

$$\#X_{\mathbb{F}_x}^{(y)}(\mathbb{F}_x^{(2)}) \geq \frac{1}{2}\#X_{\mathbb{F}_x}^R(\mathbb{F}_x^{(2)}) \geq \frac{1}{2(q^2-1)} \prod_{z \in R \cup x} (q_z - 1).$$

By [13, Prop. 5.14], if $X_F^{(y)}$ is hyperelliptic, then $X_{\mathbb{F}_x}^{(y)}$ is also hyperelliptic. Hence there is a degree-2 morphism $X_{\mathbb{F}_x}^{(y)} \to \mathbb{P}_{\mathbb{F}_x}^1$ defined over $\mathbb{F}_x$. This implies

$$\#X_{\mathbb{F}_x}^{(y)}(\mathbb{F}_x^{(2)}) \leq 2\#\mathbb{P}_{\mathbb{F}_x}^1(\mathbb{F}_x^{(2)}) = 2(q_x^2+1).$$

Comparing with the earlier lower bound on $\#X_{\mathbb{F}_x}^{(y)}(\mathbb{F}_x^{(2)})$, we get

$$\prod_{z \in R \cup x} (q_z - 1) \leq 4(q_x^2+1)(q^2-1). \tag{4.1}$$

Let $r = \sum_{z \in R} \deg(z)$. By [15, Lem.7.7], we can choose $x \notin R \cup \infty$ such that $\deg(x) \leq \log_q(r+1) + 1$. Since $\prod_{z \in R}(q_z - 1) \geq q^{r/2}$, the inequality (4.1) implies $q^{r/2} < 32q^3r$, which obviously is possible only for finitely many R. Therefore, only finitely many $X_F^{(y)}$ are hyperelliptic.

Denote by $J^{(y)}$ the Jacobian variety of $X_F^{(y)}$. To conclude the paper, we explain how one can deduce in some cases that $Ш(J^{(y)})$ is finite and has non-square order. (Of course, it is expected that Tate-Shafarevich groups are always finite.)

The definitions of the concepts discussed in this paragraph can be found in [5]. Let $\mathfrak{n} \lhd A$ be an ideal. Let $X_0(\mathfrak{n})$ be the compactified Drinfeld modular curve classifying pairs $(\phi, C_\mathfrak{n})$, where ϕ is a rank-2 Drinfeld A-modules and $C_\mathfrak{n} \cong A/\mathfrak{r}$ is a cyclic subgroup of ϕ. Let $J_0(\mathfrak{n})$ denote the Jacobian of $X_0(\mathfrak{n})_F$. Let $\Gamma_0(\mathfrak{n})$ be the level-$\mathfrak{n}$ Hecke congruence subgroup of $\mathrm{GL}_2(A)$. Let $S_0(\mathfrak{n})$ be the $\mathbb{C}$-vector space of automorphic cusp forms of Drinfeld type on $\Gamma_0(\mathfrak{n})$. Let $\mathbb{T}(\mathfrak{n})$ be the commutative $\mathbb{Z}$-algebra generated by the Hecke operators acting on $S_0(\mathfrak{n})$. The Hecke algebra $\mathbb{T}(\mathfrak{n})$ is a finitely generated free $\mathbb{Z}$-module which also naturally acts on $J_0(\mathfrak{n})$. Let $f \in S_0(\mathfrak{n})$ be a newform which is an eigenform for all $t \in \mathbb{T}(\mathfrak{n})$. Denote by $\lambda_f(t)$ the eigenvalue of t acting on f. The map $\mathbb{T}(\mathfrak{n}) \to \mathbb{C}$, $t \mapsto \lambda_f(t)$, is an algebra homomorphism; denote its kernel by I_f. The image $I_f(J_0(\mathfrak{n}))$ is an abelian subvariety of $J_0(\mathfrak{n})$ defined over F. Let $A_f := J_0(\mathfrak{n})/I_f(J_0(\mathfrak{n}))$. Similar to the case of classical modular Jacobians over $\mathbb{Q}$, the Jacobian $J_0(\mathfrak{n})$ is isogenous over F to a direct product of abelian varieties A_f, where each f is a newform of some level $\mathfrak{m}|\mathfrak{n}$ (a given A_f can appear more than once in the decomposition of $J_0(\mathfrak{r})$). This

implies that $Ш(J_0(\mathfrak{n}))$ is finite if and only if $Ш(A_f)$ is finite for all such A_f. On the other hand, by the main theorem of [10], $Ш(A_f)$ is finite if and only if

$$\mathrm{ord}_{s=1}L(A_f,s)=\mathrm{rank}_{\mathbb{Z}}A_f(F),$$

where $L(A_f,s)$ denotes the L-function of A_f; see [10] or [23] for the definition.

Let J^R denote the Jacobian of X_F^R. Let $\mathfrak{r}:=\prod_{x\in R}\mathfrak{p}_x$. The Jacquet-Langlands correspondence over F in combination with some other deep results implies that there is a surjective homomorphism $J_0(\mathfrak{r})\to J^R$ defined over F; see [18, Thm. 7.1]. Since by construction $X^{(y)}$ is a quotient of X^R, there is also a surjective homomorphism $J^R\to J^{(y)}$ defined over F. Thus, there is a surjective homomorphism $J_0(\mathfrak{r})\to J^{(y)}$ defined over F. This implies that if $Ш(J_0(\mathfrak{r}))$ is finite, then $Ш(J^{(y)})$ is also finite.

Now assume q is odd, $R=\{x,y\}$, and $\deg(x)=\deg(y)=2$. In this case $J_0(\mathfrak{r})$ is isogenous to J^R as both have dimension q^2. The dimension of $J^{(y)}$ is $(q^2-1)/2$. There are no old forms of level $\mathfrak{r}$, since $S_0(1)$, $S_0(\mathfrak{p}_x)$ and $S_0(\mathfrak{p}_y)$ are zero dimensional. Let $f\in S_0(\mathfrak{r})$ be a Hecke eigenform. The L-function $L(f,s)$ of f is a polynomial in q^{-s} of degree $\deg(x)+\deg(y)-3=1$, cf. [27, p. 227]. Hence $\mathrm{ord}_{s=0}L(f,s)\leq 1$. Using the analogue of the Gross-Zagier formula over F [22, p. 440], one concludes that $\mathrm{ord}_{s=1}L(A_f,s)\leq\mathrm{rank}_{\mathbb{Z}}A_f(F)$. The converse inequality is known to hold for any abelian variety over F; see the main theorem of [23]. Hence $Ш(A_f)$ is finite, which, as we explained, implies that $Ш(J^{(y)})$ is also finite. It remains to show that one can choose x and y so that the conditions in Theorem 4.5 hold, and therefore $Ш(J^{(y)})$ is finite and has non-square order. We need to show that one can choose x and y such that $\deg(x)=\deg(y)=2$ and $\left(\frac{\wp_y}{\mathfrak{p}_x}\right)=-1$. Fix some $x\in|F|$ with $\deg(x)=2$. Consider the geometric quadratic extension $K:=F(\sqrt{\wp_x})$ of F, and let C be the corresponding smooth projective curve over $\mathbb{F}_q$. Since $\deg(x)=2$, the genus of this curve is zero, so $C\cong\mathbb{P}^1_{\mathbb{F}_q}$. Using this observation, one easily computes that the number of places of F of degree 2 which remain inert in K is $(q^2-1)/4>0$. Thus, we can always choose $y\in|F|$ of degree 2 such that $\left(\frac{\wp_y}{\mathfrak{p}_x}\right)=-1$.

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